Set-Valued Tableaux: q-Enumeration and Catalan Combinatorics

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Joint with Sam Hopkins (Howard University) and Svante Linusson (KTH)

Preliminaries: Standard Young Tableaux

Let $\lambda \vdash n$.

$\mathcal{SYT}(\lambda) = \{\text{SYT of shape } \lambda\}$.

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$\#\mathcal{SYT}(\lambda)$ is given by the hook-length formula:

Theorem (Frame–Robinson–Thrall)

$$\#\mathcal{SYT}(\lambda) = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)},$$

where $h(u)$ is the hook-length of the cell u.

1	3	4	5	7
2	6	8		
9				

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Stanley's hook-content formula implies:

$$\sum_{T \in \mathcal{SYT}(\lambda)} q^{\mathrm{comaj}(T)} = [n]_q! \prod_{u \in \lambda} \frac{1}{[h(u)]_q},$$

where $[a]_q = 1 + q + \cdots + q^{a-1}$ and $[a]_q! = [a]_q[a-1]_q \cdots [1]_q$.

Preliminaries: (Reverse) Plane Partitions

Plane partition of λ: filling of the Young diagram of λ with pos. integers that is weakly decreasing along rows and columns.

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6	6	5
6	4	
1		

1	1	2
3	4	
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$\mathcal{PP}_m(\lambda) = \{\text{plane partitions of } \lambda \text{ with largest entry} \leq m\}$.

$\mathcal{RPP}_m(\lambda) = \{\text{reverse plane partitions of } \lambda \text{ with largest entry} \leq m\}$.

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Theorem (MacMahon)

$$\sum_{\pi \in \mathcal{PP}_m(a \times b)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{[i+j+m-1]_q}{[i+j-1]_q}.$$

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$1, 2$	$3, 5, 6$	11
$4, 7$	$8, 9, 10$	

Idea: SYT of shape λ with k additional entries.

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When $k = 1$, fillings are barely set-valued.

q-Counting BSV Objects

Theorem (Hopkins–L.–Linusson)

1. $$\sum_{S\in\mathcal{SYT}^{+1}(a\times b)} q^{\mathrm{comaj}^{+1}(S)} = \frac{[a]_q[b]_q}{[a+b]_q}[ab+1]_q \prod_{i=0}^{a-1} \frac{[i]_q!}{[b+i]_q!},$$

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comaj^{+1} = slightly intricate statistic.

The $q = 1$ version of (1) was proven by Chan, López Martín, Pflueger, and Teixidor i Bigas. (2) is a barely set-valued version of MacMahon's formula.

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(1) is the $m\to\infty$ limit of (2).

Let $T \in \mathcal{SYT}^{+k}(\lambda)$, T^* the filling of λ with only the minimal elt of each cell, and $d_1, \ldots, d_k$ the k additional elements.

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T^* breaks up into skew shapes $S_1, \ldots, S_{k+1}$, each consisting of the cells filled with the numbers $d_i + 1, \ldots, d_{i+1} - 1$.

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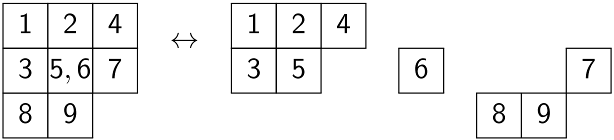

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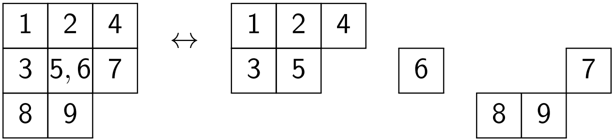

$$\mathrm{D}^{+k}(T) := \bigsqcup \mathrm{D}(S_i) \cup \{d_1, \ldots, d_k\}$$

$$\mathrm{comaj}^{+k}(T) := \sum_{i \in \mathrm{D}^{+k}(T)} n + k - i$$

$\mathcal{SYT}^{+1}(2 \times 2)$

S	$\begin{matrix} 1 & 2 \\ 3 & 4,5 \end{matrix}$	$\begin{matrix} 1 & 3 \\ 2 & 4,5 \end{matrix}$	$\begin{matrix} 1 & 2 \\ 3,4 & 5 \end{matrix}$	$\begin{matrix} 1 & 3 \\ 2,4 & 5 \end{matrix}$	$\begin{matrix} 1 & 4 \\ 2,3 & 5 \end{matrix}$
$\mathrm{D}^{+1}(S)$	$\{5\}$	$\{2,5\}$	$\{4\}$	$\{2,4\}$	$\{3\}$
$\mathrm{comaj}^{+1}(S)$	0	3	1	4	2

S	$\begin{matrix} 1 & 2,3 \\ 4 & 5 \end{matrix}$	$\begin{matrix} 1 & 2,4 \\ 3 & 5 \end{matrix}$	$\begin{matrix} 1 & 3,4 \\ 2 & 5 \end{matrix}$	$\begin{matrix} 1,2 & 3 \\ 4 & 5 \end{matrix}$	$\begin{matrix} 1,2 & 4 \\ 3 & 5 \end{matrix}$
$\mathrm{D}^{+1}(S)$	$\{3\}$	$\{4\}$	$\{2,4\}$	$\{2\}$	$\{2,3\}$
$\mathrm{comaj}^{+1}(S)$	2	1	4	3	5

Proof Outline

Proved on the level of arbitrary (finite) posets P and their sets of order ideals $\mathcal{J}(P)$.

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- Compute $\mathbb{E}_{\mu^q_{\leq,m}}(\mathrm{ddeg})$ and $\mathbb{E}_{\mu^q_{\mathrm{lin}}}(\mathrm{ddeg})$ in two different ways.

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- Specialize these results to obtain our theorem.

Two Rows

Up to now: Study $\mathcal{SYT}^{+k}$ for fixed k and let total number of entries vary.

Question: What about fixing total number of entries and letting k vary?

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Theorem (L.–Linusson)

For all $n \geq 2$

$$\left| \bigsqcup_{2b+k=n} \mathcal{SYT}^{+k}(2 \times b) \right| = \mathrm{Cat}(n-1),$$

the $n-1$st Catalan number.

Catalan^{+k} Combinatorics

Theorem (L.–Linusson)

$$\bigsqcup_{2b+k=n} \mathcal{SYT}^{+k}(2 \times b) \leftrightarrow 321 - \textit{avoiding permutations of } [n-1]$$

- *Elements of top row* $\leftrightarrow$ *right-to-left minima*
- $\#\textit{columns} = \#\{\textit{inner valleys}\} - 1$

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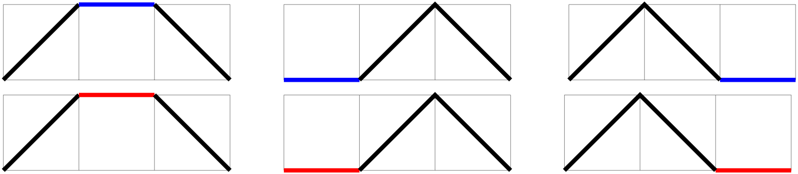
- $\#\textit{columns} = \#\{\textit{inner valleys}\} - 1$
- $$\left| \bigsqcup_{\substack{2b+k=n \\ m \text{ elts in top row}}} \mathcal{SYT}^{+k}(2\times b) \right| = \frac{1}{m}\binom{n-1}{m-1}\binom{n-2}{m-1},$$

Bicolored Motzkin Paths

Bicolored Motzkin Path: Lattice path from $(0,0)$ to $(n,0)$ using steps $\nearrow, \searrow, \rightarrow$. Steps $\rightarrow$ can be colored red or blue.



$\#\{\text{Bicolored Motzkin paths of length } n\} = \mathrm{Cat}(n+1)$ (bijection with Dyck paths of length $2n+2$)

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1. No red steps on $y = 0$
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$$\bigsqcup_{2\times b+k=n} \mathcal{SYT}^{+k}(2\times b) \leftrightarrow \mathrm{Motz}^*(n),$$

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Can also show that $\#\{\mathrm{Motz}(n) \text{ with restriction } (1)\} = \mathrm{Cat}(n)$.

Consequence: Surprising combinatorial witness of

$$\mathrm{Cat}(n-1) \leq \mathrm{Cat}(n) \leq \mathrm{Cat}(n+1)$$

Ballotlike Paths

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Theorem (L.–Linusson)

For all $0 \leq i \leq n$,

$$|\mathrm{Motz}^*(n,i)| = \binom{2n-2}{n-i-1} - \binom{2n-2}{n-i-2} + \binom{n-2}{n-i}.$$

Equivalently,

$$\left| \bigsqcup_{2b+k-i=n} \mathcal{SYT}^{+k}(b, b-i) \right| = \binom{2n-2}{n-i-1} - \binom{2n-2}{n-i-2} + \binom{n-2}{n-i}.$$

Compare with the ballot numbers $\binom{p+q}{q} - \binom{p+q}{q-1}$.

Ballotlike Paths

i/n	0	1	2	3	4	5	6	7	8
8									1
7								1	7
6							1	6	28
5						1	5	21	97
4					1	4	15	64	288
3				1	3	10	39	159	643
2			1	2	6	21	76	276	1002
1		1	1	3	9	28	90	297	1001
0	1	0	1	2	5	14	42	132	429

q-ification

Current work:

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This is a q-analog of the Catalan numbers. It seems to be new!

Question: Is there a nicer formula for it?

Determinantal formulas known for $|\mathcal{SYT}^{+k}(a \times b)|$, but the naive q-analogs **don't seem to work**.

q-ification

n	Our $q\mathrm{Cat}$
1	0
2	1
3	$q+1$
4	q^3+2q^2+q+1
5	$q^6+2q^5+3q^4+3q^3+2q^2+2q+1$
6	$q^{10}+2q^9+3q^8+7q^7+6q^6+5q^5+6q^4+7q^3+3q^2+q+1$

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Thank you!

Merci beaucoup!

q-Toggle-Symmetry

Let P be a finite poset, $q > 0$. For $p \in P$ the toggle statistics are

$$\mathcal{T}_p^+(I) = \begin{cases} 1, & I \cup \{p\} \in \mathcal{J}(P) \\ 0 & \text{else} \end{cases} \qquad \mathcal{T}_p^-(I) = \begin{cases} 1, & p \in \max(I) \\ 0 & \text{else} \end{cases}$$

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The q-togglability statistic is $\mathcal{T}_p^q := \mathcal{T}_p^+ - q\mathcal{T}_p^-$.

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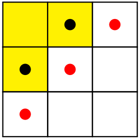

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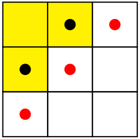

μ is q-toggle-symmetric if $\mathbb{E}_\mu(\mathcal{T}_p^q) = 0$ for all p, that is, we are q times as likely to toggle a p out of a random $I \in \mathcal{J}(P)$ as we are to toggle p into a random $I \in \mathcal{J}(P)$.